ENG SODDA RATSIONAL KASRLAR VA ULARNI INTEGRALLASH

Akbarova Shaxlo Sodikovna
O'zbekiston Respublikasi Ichki ishlar vazirligining 2-sonli Toshkent akademik litseyi matematika fani o'qituvchisi

Annotatsiya: *Ko'p yillik pedagogik tajribadan shu narsa ma'lumki, o'quvchilar kasr ratsional funksiyalarni integrallashga oid bilim va ko'nikmalarni egallashda ancha qiynaladi. Ana shuni nazarda tutib ushbu maqolada o'quvchilaming kasr ratsional funksiyalarni integrallashlari uchun oson hamda qulay usullarini ko'rsatib o'tilgan.*

Kalit so'zlar: *Kasr ratsional funksiyalar, to'g'ri va noto'g'ri kasr ratsional funksiyalar, ko'p hadni ko'p hadga bo'lish, sodda kasrlar, ratsional funksiya yoyilmasi, noma'lum koeffitsientlar usuli.*

Ratsional kasr funksiyalarni integrallash

To'g'ri va noto'g'ri kasr ratsional funksiyalar haqida.

Yuqorida ko'rsatilgan integrallash usullari yordamida hamma integrallarni hisoblash mumkin deb bo'lmaydi. Shunday funksiyalar sinflari borki, ular uchun muayyan usullardan foydalanib ularni jadval integrallariga yoki integrallash usullaridan foydalanish uchun qulay qo'lga keltirish mumkin, shunday funksiya sinflaridan ayrimlarini qaraymiz.

Ma'lumki, har qanday ratsional funksiyani ushbu ko'rinishida ifodalash mumkin, ya'ni

$$\frac{Q(x)}{P(x)}=\frac{b_0x^m+b_1x^{m-1}+...+b_m}{a_0x^n+a_1x^{n-1}+...+a_n}$$

Suratdagi ko'phadning darajasi mahrajdagi ko'phad darajasidan kichik, ya'ni $m<n$ bo'lsa, berilgan kasrga to'g'ri kasr ratsional funksiya deyiladi. Suratdagi ko'phadning darajasi $m\geq n$ bo'lsa, noto'g'ri kasr ratsional funksiya deyiladi. Kasr noto'g'ri kasr ratsional funksiya bo'lsa, suratni mahrajga, ko'phadni ko'phadga bo'lish qoidasiga asosan bo'lib, uning butun qismini ajratib, uni butun va to'g'ri kasr ratsional funksiyaga keltirish mumkin.

Masalan, $\frac{x^3+3x^2+3x+1}{x^2-x}$ noto‘g‘ri kasr ratsional funksiyani, x^3+3x^2+3x+1 ko‘phadni x^2-x ikki hadga bo‘lib, $\frac{x^3+3x^2+3x+1}{x^2-x}=x+4+\frac{7x+1}{x^2-x}$ ko‘rinishda yozish mumkin. Umumiy holda, $\frac{Q(x)}{P(x)}$ noto‘g‘ri kasr ratsional funksiya bo‘lsa, uni $\frac{Q(x)}{P(x)}=T(x)+\frac{R(x)}{P(x)}$ shaklda ifodalash mumkin, bu yerda $T(x)$ butun ratsional funksiya, $\frac{R(x)}{P(x)}$ to‘g‘ri ratsional kasr funksiyadan iborat. $T(x)$ funksiyani osongina integrallash mumkin.

Sunday qilib, noto‘g‘ri kasr ratsional funksiyani integrallashni, $\frac{R(x)}{P(x)}$ to‘g‘ri kasr ratsional funksiyani integrallashga keltiriladi.

To‘g‘ri kasr ratsional funksiyalarni soda kasrlar ko‘rinishida ifodalash va ularni integrallash

1) $\frac{A}{x-a}$; 2) $\frac{A}{(x-a)^k}$ (*k> 1butun son); 3)* $\frac{Ax+B}{x^2+px+q}$ ($\frac{p^2}{4}-q<0$ ya’ni, kvadrat uchhad haqiqiy ildizga ega emas); 4) $\frac{Ax+B}{(x^2+px+q)^n}$ ($n>1$ butun son, $\frac{p^2}{4}-q<0$) ratsional to‘g‘ri kasrlarga *sodda kasr ratsional funksiyalar deyiladi*. (*A,B,p,q,a*-haqiqiy sonlar).

Birinchi ikki xildagi funksiyalarni osongina integrallash mumkin, ya’ni,

1) $\int\frac{A}{x-a}dx=A\ln|x-a|+C$ bo‘ladi.

2) $\int\frac{A}{(x-a)^k}dx=A\int(x-a)^{-k}d(x-a)=A\frac{(x-a)^{-k+1}}{-k+1}+C=\frac{A}{1-k}\cdot\frac{1}{(x-a)^{k-1}}+C$

Endi ushbu 3) $\int\frac{Ax+B}{x^2+px+q}dx$ integralni hisoblaymiz.

Oldin xususiy hol $\int\frac{1}{x^2+px+q}dx$ integralni qaraylik. x^2+px+q dan to‘la kvadrat ajratib, $x+\frac{p}{2}=t$ almashtirishdan keyin quyidagini hosil qilamiz:

$\int \frac{1}{x^2+px+q}dx = \int \frac{1}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}dx = \left| \begin{matrix} x+\frac{p}{2}=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{\left(t^2+a^2\right)}$, bu yerda $a=\sqrt{q-\frac{p^2}{4}}$.

Oxirgi integralda jadval integralidan foydalanib, $\int \frac{1}{x^2+px+q}dx = \frac{1}{a}arctg\frac{t}{a}+C = \frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C$ natijani hosil qilamiz. Endi $\int \frac{Ax+B}{x^2+px+q}dx$ integralni hisoblaymiz.

$Ax+B=(2x+p)\frac{A}{2}-\frac{Ap}{2}+B$ shakl o‘zgartirishdan foydalanib, integralni quyidagicha yozamiz:

$$\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{(2x+p)\frac{A}{2}-\frac{Ap}{2}+B}{x^2+px+q}dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \left(B-\frac{Ap}{2}\right)\int \frac{1}{x^2+px+q}dx.$$

Oxirgi tenglikni o‘ng tomonidagi birinchi integral $\int \frac{2x+p}{x^2+px+q}dx = \int \frac{d\left(x^2+px+q\right)}{x^2+px+q} = \ln\left|x^2+px+q\right|+C_1$ bo‘lib, ikkinchi integral formulaga asosan, $\int \frac{dx}{x^2+px+q} = \frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C_2$. Shunday qilib, $\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln\left|x^2+px+q\right| + \frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C$ natijaga ega bo‘lamiz.

Bir nechta misollar qaraymiz:

1-misol. $\int \frac{x^4}{x^2+9}dx$ integralni hisoblang.

Yechish: Integral ostidagi funksiya noto‘g‘ri kasr ratsional funksiyadan iborat. Uning butun qismini ajratganimizda $\frac{x^4}{x^2+9} = x^2-9+\frac{81}{x^2+9}$ hosil bo‘ladi.

Shunday qilib,

$$\int \frac{x^4}{x^2+9}dx = \int \left(x^2-9+\frac{81}{x^2+9}\right)dx = \frac{x^3}{3}-9x+81\frac{1}{3}arctg\frac{x}{3}+C = \frac{x^3}{3}-9x+27arctg\frac{x}{3}+C.$$

2-misol. $\int \frac{x+3}{x^2-8x+25}dx$ integralni hisoblang.

Yechish: Maxrajdagi kvadrat uchhaddan to‘la kvadrat ajratamiz: $x^2-8x+25 = x^2-8x+16-16+25 = \left(x-4\right)^2+9$, hamda $x-4=t,\ dx=dt$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int\frac{x+3}{x^2-8x+25}dx=\int\frac{t+4+3}{t^2+9}dt=\int\frac{tdt}{t^2+9}+7\int\frac{dt}{t^2+9}=\frac{1}{2}\int\frac{2tdt}{t^2+9}+7\int\frac{dt}{t^2+3^2}=\frac{1}{2}\ln\left|t^2+9\right|+\frac{7}{3}arctg\frac{t}{3}+C=$$

$\frac{1}{2}\ln\left(x^2-8x+25\right)+\frac{7}{3}arctg\frac{x-4}{3}+C$. $\frac{R(x)}{P(x)}$ to'g'ri kasr ratsional funksiyaning mahrajini $P(x)=(x-a)^r\cdot(x-b)^s\cdot...\cdot\left(x^2+2px+q\right)^t\cdot\left(x^2+2kx+e\right)^m\cdot...,$ ko'rinishda ifodalash mumkin bo'lsa, bu funksiyani yagona

$$\frac{R(x)}{P(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+...+\frac{A_r}{(x-a)^r}+\frac{B_1}{(x-b)}+...+\frac{B_s}{(x-b)^s}+...+\frac{M_1x+N_1}{x^2+2px+q}+...+$$

$$+\frac{M_tx+N_t}{\left(x^2+2px+q\right)^t}+\frac{F_1x+E_1}{\left(x^2 2kx+e\right)}+...+\frac{F_mx+E_m}{\left(x^2 2kx+e\right)^m}+...$$ ko'rinishda yozish mumkin.

Bunda *r,s,…,t,m*- musbat butun sonlar, *a,b,p,q,k,e* –haqiqiy sonlar.

$A_1,A_2,...,A_r,B_1,...,B_s,M_1,N_1,...,M_t,N_t,...$ lar ayrim haqiqiy sonlar

$$\frac{R(x)}{P(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+...+\frac{A_r}{(x-a)^r}+\frac{B_1}{(x-b)}+...+\frac{B_s}{(x-b)^s}+...+\frac{M_1x+N_1}{x^2+2px+q}+...+$$

$$+\frac{M_tx+N_t}{\left(x^2+2px+q\right)^t}+\frac{F_1x+E_1}{\left(x^2 2kx+e\right)}+...+\frac{F_mx+E_m}{\left(x^2 2kx+e\right)^m}+...$$ tenglikka to'g'ri ratsional funksiyaning *sodda kasrlar orqali yoyilmasi* deyiladi.

Yoyilmadagi $A_1,A_2,...,A_r,M_1,N_1,...,M_t,N_t,...$ koeffitsiyentlarni toppish uchun uni $P(x)$ ga ko'paytiramiz. $P(x)$ ko'phad bilan $A_1,A_2,...,A_r,B_1,...,B_s,M_1,N_1,...,M_t,N_t,...$ yoyilmaning o'ng tomonida hosil bo'lgan ko'phad o'zaro teng bo'lishi uchun bir hil darajali x lar koeffitsiyentlari o'zaro teng bo'lishi kerak. Bir xil darajali x lar koeffitsiyentlarini tenglashtirib $A_1,A_2,...,A_r,M_1,N_1,...,$ noma'lum koeffitsiyentlarga nisbatan chiziqli tenglamalar sistemasini hosil qilamiz. Bu tenglamalar sistemasini yechib aniqmas koeffitsiyentlarni topamiz.

Ratsional funksiya yoyilmasidagi noma'lum koeffitsiyentlarni bunday usul bilan topishga *noma'lum koeffitsiyentlar usuli* deyiladi. Bu usulni quyidagi misol orqali ko'rib chiqamiz:

Misol: $\int\frac{dx}{(x-2)(x-3)}$ integralni hisoblang.

Yechish: $\frac{1}{(x-2)(x-3)}$ fuksiyani $\left(\frac{-1}{x-2}+\frac{1}{x-3}\right)$ ikkita soda kasrning ayirmasi ko'rinishida yozish mumkin. Shuning uchun, $\int\frac{dx}{(x-2)(x-3)}=-\int\left(\frac{1}{x-2}-\frac{1}{x-3}\right)dx=-\ln|x-2|+\ln|x-3|+C=\ln\left(\frac{x-3}{x-2}\right)+C$ bo'ladi.

Mustahkamlash uchun savollar:

1. Butun ratsional funksiya nimadan iborat?
2. Kasr ratsional funksiyalar qanday?
3. To'g'ri va noto'g'ri kasr ratsional funksiyalar deb nimaga aytiladi?
4. Noto'g'ri kasr ratsional funksiyani integrallash, to'g'ri ratsional funksiyani integrallashga qanday qilib keltiriladi?
5. Sodda kasr ratsional funksiyalar deb nimaga aytiladi?
6. Sodda kasr ratsional funksiyalar qanday integrallanadi?
7. Aniqmas koeffitsiyentlar usuli nima?

Mustaqil bajarish uchun topshiriqlar:

Ushbu ratsional funksiyalarni integrallang:

1) $\int\frac{x+2}{x^3-2x^2}dx$

2) $\int\frac{5x-1}{2x^2+x-3}dx$

3) $\int\frac{2x+5}{(x-4)(x+5)}dx$

4) $\int\frac{6x-7}{x^3-4x^2+4x}dx$

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