# ZYGMUND TYPE INEQUALITIES FOR DOUBLE SINGULAR CAUCHY-STIELTJES INTEGRAL

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**Abstract.** *For the double singular Cauchy-Stiltjes integral over spanning set of the bicylindric domain, the Zygmund type estimate is obtained that relates the partial and mixed moduli of continuity of the singular integral and its density. On this basis, some spaces are constructed that are invariant with respect to the double singular integral.*

**Key words**: *Zygmund estimate, double singular integral, partial and mixed modulus of continuity, invariant spaces.*

Let $\gamma^k$ be a closed Jordan rectifiable curve (c.j.r.c) on the complex plane $z_k$ ($k$ = 1,2), which divides the complex plane into two parts the interior $D_k^+$ and the exterior $D_k^-$. The curves $\gamma^1$ and $\gamma^2$ define four bicylindric domains $D^{\pm} = D_1^{\pm} \times D_2^{\pm}$ with the boundaries having the common part $\Delta = \gamma^1 * \gamma^2$ known as spanning set. Let

$$\Phi_\psi(z) = \frac{1}{(2\pi i)^2} \int_\Delta \frac{f(s)d\psi(s)}{\prod_{k=1}^{2}(s_k - z_k)} \qquad (1)$$

be the double Cauchy-Stiltjes type integral, where $z = (z_1, z_2)$, $s = (s_1, s_2)$, $d\psi(s) = d\psi_1(s)d\psi_2(s)$, $f(s) \in C_\Delta$, $C_\Delta$ is the space of continuous functions on $\Delta$, $\psi_k(s)$ being functions of bounded variation on $\gamma^k$ $(k = 1,2)$. Under the investigation of limiting values of the function $\Phi_\psi(z)$ there appear the following singular integrals:

$$g_\psi^{1,1}(t) = \int_\Delta \frac{\left(\overset{1,1}{\Delta} f\right)(s;t)d\psi(s)}{\prod_{k=1}^{2}(s_k - z_k)},\; g_\psi^{1,0}(t) = \int_{\gamma^1} \frac{\left(\overset{1,0}{\Delta} f\right)(s;t)d\psi(s_1)}{s_1 - t_1},\; g_\psi^{0,1}(t) = \int_{\gamma^2} \frac{\left(\overset{0,1}{\Delta} f\right)(s;t)d\psi(s_2)}{s_2 - t_2},$$

(2)

where $\Delta f(s;t) = f(s_1, s_2) - f(s_1, t_2) - f(t_1, s_2) + f(t_1, t_2)$, $\Delta f(s_{t_2};t) = f(s_1, t_2) - f(t_1, t_2)$, $\Delta f(s_{t_1};t) = f(t_1, s_2) - f(t_1, t_2)$.

We denote

$$\tilde{f}_{\psi}(t)=g_{\psi}^{1,1}(t)+g_{\psi_1}^{1,0}(t)+g_{\psi_2}^{0,1}(t). \qquad (3)$$

In the case $\psi_i(t)=t\ (i=1,2)$, we write $\tilde{f}(t)=g^{1,1}(t)+g^{1,0}(t)+g^{0,1}(t)$.

To study the properties of the integral (2), we arrive at the need to select the following main characteristics of the functions $f\in C_{\Delta}$:

1) mixed modulus of continuity ($\delta=(\delta_1,\delta_2)$, $\delta_1>0$, $\delta_2>0$, $\xi=(\xi_1,\xi_2)$):

$$\overset{1,1}{\omega}_f(\delta)=\delta_1\cdot\delta_2\sup_{\xi_1\ge\delta_1,\xi_2\ge\delta_2}\frac{\overset{1,1}{\omega}(f:\xi_1,\xi_2)}{\xi_1\xi_2}=\delta\sup_{\xi\ge\delta}\frac{\overset{1,1}{\omega}(f:\xi)}{\xi}\text{, where }\overset{1,1}{\omega}(f,\delta)=\sup_{\substack{|s_1-t_1|<\delta_1\\|s_2-t_2|<\delta_2}}\left|\left(\overset{1,1}{\Delta}f\right)(s;t)\right|;$$

2) modulus of partial continuity $\overset{1,0}{\omega}_f(\delta_1)=\delta_1\sup\limits_{\xi_1\ge\delta_1}\dfrac{\overset{1,0}{\omega}(f,\xi_1)}{\xi_1}$, $\overset{1,0}{\omega}(f;\delta_1)=\sup\limits_{\substack{t_2\in\gamma^2\\|s_1-t_1|\le\delta_1}}\left|\Delta f\left(s_{t_2};t\right)\right|$

and $\overset{0,1}{\omega}_f(\delta_2)=\delta_2\sup\limits_{\xi_2\ge\delta_2}\dfrac{\overset{0,1}{\omega}(f,\xi_2)}{\xi_2}$, $\overset{0,1}{\omega}(f;\delta_2)=\sup\limits_{t_1\in\gamma^1}\sup\limits_{|s_2-t_2|\le\delta_2}\left|\Delta f\left(s_{t_1};t\right)\right|$.

Let us $\Phi_{(0,d_1]\times(0,d_2]}=\Phi_{T^2}$ denote the set of functions $\omega(\delta_1,\delta_2)=\omega(\delta)$ defined on $T^2=(0,d_1]\times(0,d_2]$ and belonging to $\Phi^1$ in each argument, i.e.

1) $\omega(\delta)\in\Phi^1_{(0,d_2]}$ by $\delta_2$ for any fixed $\delta_1$;

2) $\omega(\delta)\in\Phi^1_{(0,d_1]}$ by $\delta_1$ for any fixed $\delta_2$.

We introduce the Zygmund type operator

$$Z\left(\omega;\delta,\theta^{\psi},\breve{\theta}^{\psi}\right)=Z\left(\omega;\delta_1,\delta_2,\theta_1^{\psi_1},\theta_2^{\psi_2},\breve{\theta}_1^{\psi_1},\breve{\theta}_2^{\psi_2}\right)=\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1d\xi_2+$$

$$+\delta_1\int_{\theta_1^{\psi_1}(\delta_1)}^{d_1}\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1d\xi_2+\delta_2\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_1d\xi_2+$$

$$+\delta_1\delta_2\int_{\theta_1^{\psi_1}(\delta_1)}^{d_1}\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2\cdot\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_1d\xi_2.$$

**Theorem 1**. Let $f\in C_{\Delta}$, $\psi_k\in V_{\gamma}\ (k=1,2)$. If

$$\int_0^d\frac{\overset{1,2}{\omega}_f(\eta)}{\eta}d\theta^{\psi}(\eta)\overset{def}{=}\int_0^{d_1}\int_0^{d_2}\frac{\overset{1,2}{\omega}_f(\eta_1,\eta_2)}{\eta_1\eta_2}d\theta_1^{\psi_1}(\eta_1)d\theta_2^{\psi_2}(\eta_2)<\infty$$

then the limits $\lim\limits_{\varepsilon_1\to0}g_{\psi,\varepsilon}^{1\cdot2}(t_1,t_2)$, $\lim\limits_{\varepsilon_2\to0}g_{\psi,\varepsilon}^{1\cdot2}(t_1,t_2)$, $\lim\limits_{\substack{\varepsilon_1\to0\\\varepsilon_2\to0}}g_{\psi,\varepsilon}^{1\cdot2}(t_1,t_2)$ with any fixed $\varepsilon_2\in(0,d_2]$ in the first limit and any fixed $\varepsilon_1\in(0,d_1]$ in the second one, exist uniformly in $t_1,t_2$.

In the following theorem we use the notation

$$\Im_0^{\psi}(\Delta)=\left\{\ f\in C_{\Delta}:\int_0^d\frac{\overset{1,2}{\omega}_f(\xi)}{\xi}d\theta^{\psi}(\xi)<\infty,\ \int_0^{d_1}\frac{\overset{1,0}{\omega}_f(\xi_1)}{\xi_1}d\theta_1^{\psi_1}(\xi_1)<\infty,\ \int_0^{d_2}\frac{\overset{0,2}{\omega}_f(\xi_2)}{\xi_2}d\theta_2^{\psi_2}(\xi_2)<\infty\ \right\}.$$

**Theorem 2.** Let $f \in \Im_0^{\psi}(\Delta)$ with $\psi = (\psi_1, \psi_2)$, $\psi_k \in V_k$, $k$=1,2. Then the following inequalities are valid

$$\overset{1,1}{\omega}_{\bar{f}}(\delta_1, \delta_2) \le C_1 Z\left(\overset{1,1}{\omega}_f ; \delta; \theta^{\psi}; \breve{\theta}^{\psi}\right),\ 0 < \delta_k \le d_k,\ k=1,2$$

$$\overset{1,0}{\omega}_{\bar{f}}(\delta_1) \le C_2\left[Z\left(\overset{1,1}{\omega}_f ; \delta_1, d_2, \theta^{\psi}, \breve{\theta}^{\psi}\right) + Z\left(\overset{1,0}{\omega}_f ; \delta_1, \theta_1^{\psi_1}, \breve{\theta}_1^{\psi_1}\right)\right],\ 0 < \delta_1 \le d_1,$$

$$\overset{0,1}{\omega}_{\bar{f}}(\delta_2) \le C_3\left[Z\left(\overset{1,1}{\omega}_f ; d_1, \delta_2, \theta^{\psi}, \breve{\theta}^{\psi}\right) + Z\left(\overset{0,1}{\omega}_f ; \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2}\right)\right],\ 0 < \delta_2 \le d_2. \qquad (4)$$

Let $\omega \in \Phi_{T^2}$. We introduce the linear space

$$K_\omega = \left\{\ f \in C_\Delta : \overset{1,2}{\omega}_f(\delta_1, \delta_2) = O(\omega(\delta_1, \delta_2)),\ \overset{1,0}{\omega}_f(\delta_1) = O(\omega(\delta_1, d_2)),\ \overset{0,2}{\omega}_f(\delta_2) = O(\omega(d_1, \delta_2))\ \right\}$$

and equip it with the norm

$$\|f\|_{K_\omega} = \max\left\{\overset{1,2}{C}_f, \overset{1,0}{C}_f, \overset{0,2}{C}_f, \|f\|_{C_\Delta}\right\},$$

where

$$\overset{1,2}{C}_f = \sup_{\delta_1, \delta_2 > 0} \frac{\overset{1,2}{\omega}_f(\delta_1, \delta_2)}{\omega(\delta_1, \delta_2)},\ \overset{1,0}{C}_f = \sup_{\delta_1 > 0} \frac{\overset{1,0}{\omega}_f(\delta_1)}{\omega(\delta_1, d_2)},\ \overset{0,2}{C}_f = \sup_{\delta_2 > 0} \frac{\overset{0,2}{\omega}_f(\delta_2)}{\omega(d_1, \delta_2)}.$$

**Theorem 3.** Let $d\psi(t) = F(t)dt$, where $F(t)$ is limiting value of the function analytic in $D^{\pm}$ and continuous up to the boundary and

$$\omega \in \Im_0\Phi = \left\{\ \omega \in \Phi_{T^2} : \int_0^d \frac{\omega(\xi)}{\xi} d\theta^F(\xi) < \infty,\right.$$

$$\left.\int_0^{d_1} \frac{\omega(\xi_1, d_2)}{\xi_1} d\theta_1^F(\xi_1) < \infty,\ \int_0^{d_2} \frac{\omega(d_1, \xi_2)}{\xi_2} d\theta_2^F(\xi_2) < \infty\ \right\}$$

Then $f \in K_\omega \Rightarrow \tilde{f}_F \in K_z(\omega, \delta, \theta^F)$.

**Theorem 4**. Let $f \in K_\omega$. Then

$$\theta_k^F(\delta) \sim \delta\ (k = 1,2),\quad \omega \in \{\omega \in \Im_0\Phi :\ Z(\omega; \delta_1, \delta_2) = O(\omega(\delta_1, \delta_2))\}$$

$\tilde{f} \in K_\omega$ as well and $\|\tilde{f}\|_{|K_\omega|} \le \|f\|_{|K_\omega|}$.

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